Dependence of intermittency scaling on threshold in chaotic systems

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Numerical and experimental investigations of intermittency in chaotic systems often lead to claims of universal classes based on the scaling of the average length of the laminar phase with parameter variation. We demonstrate that the scaling in general depends on the choice of the threshold used to define a proper laminar region in the phase space. For sufficiently large values of the threshold, the scaling exponent tends to converge but significant fluctuations can occur particularly for continuous-time systems. Insights into the dependence can be obtained using the idea of Poincaré recurrence.

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An extremely common phenomenon in nonlinear dynamical systems arising from a variety of disciplines is intermittency [1]. By definition, intermittency describes the situation where a dynamical variable of the system exhibits two distinct types of characteristically different behaviors interspersed in the course of time evolution. For example, the system can exhibit nearly regular motion for an extended period of time and then undergoes a temporal transition to a highly irregular state before returning to the regular state, while the transitions can occur at random times. In this case, the time series of a dynamical variable contains phases of regular motion (laminar phases) interrupted by (often) short phases of irregular bursts (turbulent phases). In low-dimensional dynamical systems, the seminal work of Pomeau and Manneville established the existence of three types of intermittency [1], named simply as type-I, type-II, and type-III intermittency, which are generated by saddle-node, Hopf, and inverse periodic-doubling bifurcations, respectively. Another common type of intermittency is crisis-induced intermittency, discovered by Grebogi *et al.* [2], where the two distinct phases of motion are both chaotic but with different characteristics. In dynamical systems with symmetry, on-off intermittency can occur, where the "off" state typically indicates that the motion of the system is restricted to some low-dimensional invariant subspace as determined by the symmetry and the "on" state characterizes motion away from the invariant subspace [3]. Intermittency is also common in spatially extended dynamical systems [4].

Because of the ubiquity of intermittency in all kinds of nonlinear dynamical systems, the observation and characterization of intermittency have become a basic tool to probe the underlying system. This is especially true in experimental studies, where the equations of the system under study are often unknown. Insights into the system dynamics can be obtained by establishing scaling laws associated with any observed intermittency. In this regard, a common practice is to search, from given intermittent signals measured from the system, for scaling between the average duration of the laminar phase and system parameter variation beyond the onset of intermittency [1,2]. To achieve this goal, one often sets a threshold and regards the system as being in the laminar phase if the dynamical variable of interest stays below the threshold. A large number of time intervals for the laminar motion can then be accumulated, yielding an average length, say $\langle l\rangle$, of the laminar phase. The value of $\langle l\rangle$ typically depends on system parameters. Let p be a bifurcation parameter of the system and let p_c denote the critical parameter value for the onset of the intermittency behavior. For p near p_c, the following algebraic scaling law is typically observed for intermittency:

$$\langle l\rangle \sim |p-p_c|^{-\alpha}, \tag{1}$$

where $\alpha > 0$ is an algebraic scaling exponent. For type-I (II, III) intermittency, the theoretical value of α is 1/2 (1,1). For crisis-induced intermittency, the value of α is determined by the eigenvalues of the unstable periodic orbit mediating the crisis [2]. When scaling law (1) is observed, the system is regarded as belonging to a certain universal class of dynamical systems exhibiting intermittency.

When the algebraic exponent is extrapolated from an experimentally or numerically obtained scaling in the form of Eq. (1), a practical issue is how to choose a proper threshold for classifying the motion as being laminar or bursting. A difficulty is that the value of the extracted scaling exponent often depends on the threshold. By adjusting the threshold, one can in principle obtain a range of values for the exponent. A natural question thus concerns about the dependence of the exponent on the threshold. We are somewhat surprised to find that, despite the tremendous amount of work on intermittency in the past, there has been no systematic investigation of this fundamental issue in applied nonlinear dynamics. The purpose of this Brief Report is to address this issue. In particular, we shall adopt the concept of Poincaré recurrence, which has recently been exploited to detect various transitions to chaotic synchronization [5] and to investigate transient chaos in leaky dynamical systems [6]. To define recurrences for a dynamical system exhibiting intermittency, we divide the phase space into two regions, one hosting laminar motion and other corresponding to bursting (or "turbulent") motion. The average duration of the laminar phase can be related to the average recurrent time to the turbulent region denoted by T_t. Let $\langle t\rangle$ be the average duration of the turbulent phase and T_l be the average recurrent time to the

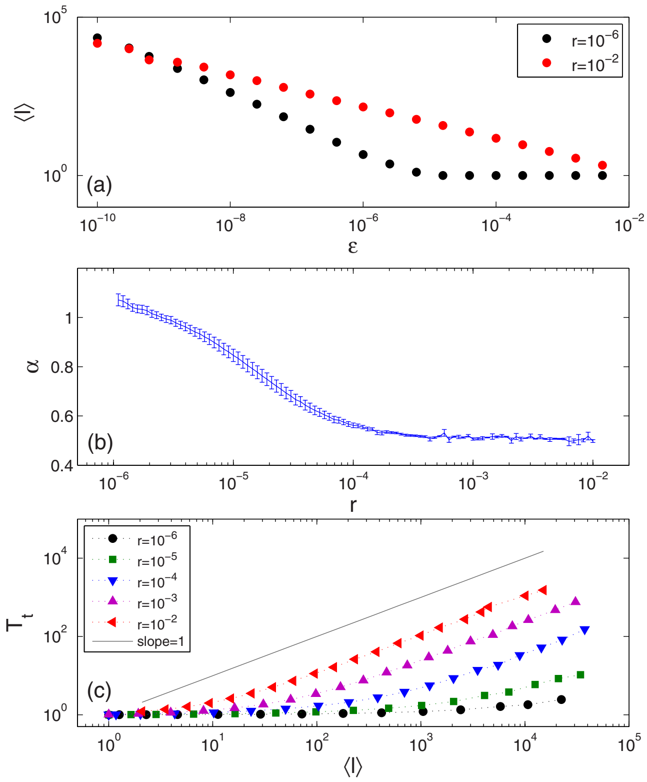

FIG. 1. (Color online) For the logistic map, (a) algebraic scaling of the average length $\langle l\rangle$ of the laminar phase with the parameter variation ε for two different values of the threshold r ($r=10^{-6}$ and 10^{-2}, respectively), (b) dependence of the scaling exponent α on the threshold, and (c) relation between recurrence time T_t and $\langle l\rangle$ for different values of r.

laminar region. A natural-measure-based consideration leads to the following relation:

$$\frac{\langle l\rangle}{\langle t\rangle}=\frac{T_t}{T_l}. \tag{2}$$

We shall present evidence that, when proper recurrent regions are chosen, the quantities $\langle t\rangle$ and T_l usually do not vary appreciably with parameter but T_t does. In this case, the scaling of $\langle l\rangle$ with parameter variation is mainly determined by the scaling of T_t. Our key point is that the recurrence time T_t typically depends on the size of the neighborhood used to define T_l and T_t in the first place. Consequently, the average length of the laminar phase will also depend on the size or, equivalently, on the threshold that defines the laminar phase. In general, we expect a strong dependence of the algebraic scaling exponent α on the threshold. While convergence of α usually occurs for sufficiently large values of the threshold, large fluctuations in the exponent tend to occur, particularly for continuous-time dynamical systems. The implication is that, in numerical or experimental situations, any scaling of the average length of the laminar phase with parameter variation should be interpreted with caution.

To be concrete, we consider type-I intermittency, which appears through a saddle-node bifurcation. Before the occurrence of intermittency, in the phase space there is a periodic attractor and an unstable periodic orbit of the same period. As the parameter p passes through p_c, the two orbits coalesce and disappear simultaneously leaving behind a "weakly unstable" phase-space region around the original attractor where trajectories can spend long stretches of time. This defines the laminar phase. Since the region is not attracting, a trajectory temporally confined in it can exit, giving rise to a bursting phase. If the dynamics in the region can be described by a one-dimensional map of the type $x_{n+1}=x_n+ax_n^z+\varepsilon$, where a and ε are parameters, the algebraic scaling exponent in Eq. (1) can be obtained as $\alpha=(z-1)/z$, provided that the size of the neighborhood is chosen properly [1]. For the typical case of $z=2$, one obtains the "universal" exponent $\alpha=1/2$. This one-dimensional argument is, however, difficult to extend to higher-dimensional systems.

We shall examine intermittency in dynamical systems of any dimension using the idea of Poincaré recurrence. Before the occurrence of intermittency, say, for $p<p_c$, there is a periodic attractor in the phase space. For simplicity, we assume it is a fixed-point attractor $\mathbf{x}_c$, and its neighborhood can be defined as $N=\{\mathbf{x}|\|\mathbf{x}-\mathbf{x}_c\|<r\}$, where r is its size. Let $\varepsilon\equiv p-p_c$. For $\varepsilon\lesssim 0$, the natural measure of the neighborhood is $\mu_N(r,\varepsilon)=1$, which is independent of the values of ε and r, insofar they are small. For $\varepsilon\gtrsim 0$, intermittency arises and the underlying attractor becomes chaotic (the intermittency route to chaos [1]). As a result, μ_N depends on both ε and r. Since the laminar phase is nothing but motion inside the neighborhood, the probability for observing a laminar phase, denoted by μ_l, is μ_N. As ε is increased, the probability for bursting motion increases, so μ_N decreases for a fixed value of r. For $p\gtrsim p_c$ ($\varepsilon\gtrsim 0$), we can write

$$\mu_l=1-O(\varepsilon^{\beta}), \tag{3}$$

where $\beta>0$ is a constant. We also observe that, for any fixed value of ε, as r is increased, μ_l should increase. In fact, we have $\mu_l\sim r^D$, where D is the pointwise dimension for typical points on the attractor, which is equal to the Lyapunov dimension of the attractor [6,7]. Since the phase-space region hosting the turbulent motion is complementary to the small neighborhood where laminar motion occurs, the measure of the turbulent phase is

$$\mu_t=1-\mu_l=O(\varepsilon^{\beta}). \tag{4}$$

Based on the natural measures, we can define the average recurrence time. In particular, the neighborhood corresponding to the laminar phase will be visited by a typical trajectory from time to time, say at $\tau_1,\tau_2,\ldots,\tau_n$. The first recurrence time to the laminar neighborhood is then $t_i=\tau_i-\tau_{i-1}$ ($\tau_0\equiv 0$), and the average recurrence time is given by $T_l=\lim_{n\to\infty}(1/n)\Sigma_{i=1}^{n}t_i$. Similarly, we can define the average recurrence time to the turbulent phase denoted by T_t. Kac's lemma [8], which relates the average recurrence time to the natural measure, gives

$$T_{l,t}=\frac{1}{\mu_{l,t}}. \tag{5}$$

We then have

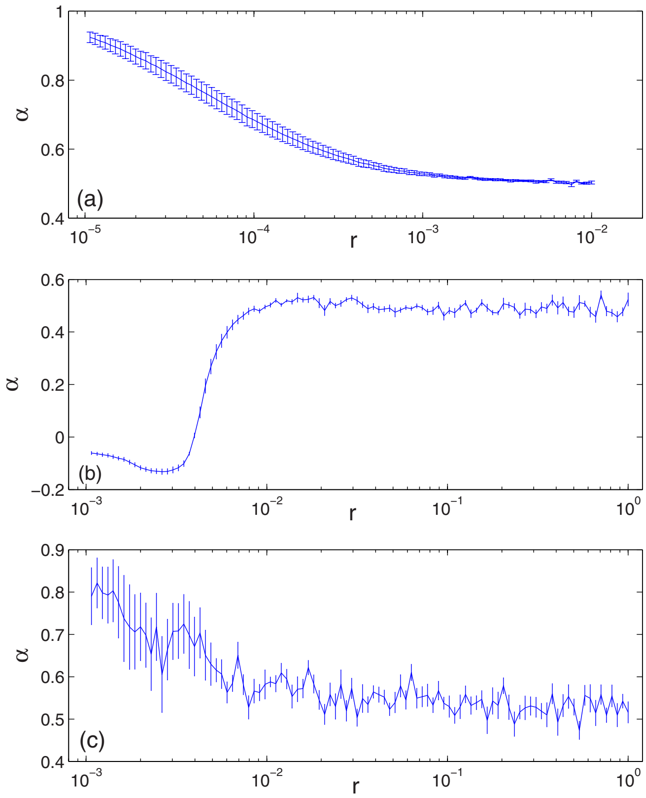

FIG. 2. (Color online) Dependence of the scaling exponent α on the threshold for (a) Hénon map, (b) classical Lorenz system, and (c) generalized Lorenz system.

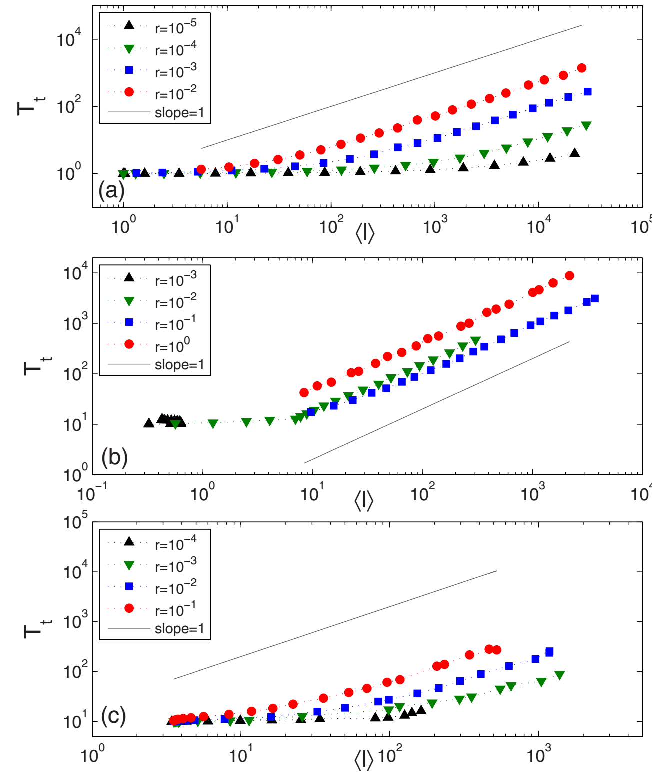

FIG. 3. (Color online) Relation between T_t, recurrent time to the turbulent phase, and $\langle l\rangle$ for (a) Hénon map, (b) classical Lorenz system, and (c) generalized Lorenz system.

$$T_l = \frac{1}{\mu_l} = \frac{\mu_l + \mu_t}{\mu_l} = \frac{Q_l + Q_t}{Q_l} = \frac{Q_l T + Q_t T}{Q_l T} = \frac{\langle l\rangle n_l + \langle t\rangle n_t}{\langle l\rangle n_l} \approx \frac{\langle l\rangle + \langle t\rangle}{\langle l\rangle}, \tag{6}$$

where T is the total observation time, and Q_l and Q_t are the portions of time the system spends in the laminar and the turbulent phase, respectively. The quantities n_l and n_t denote the numbers of laminar and turbulent phases contained within T, respectively. Due to ergodicity, we can identify $\mu_{l,t}$ with $Q_{l,t}$. Since $n_l - n_t = \pm 1$ and for large T, $n_{l,t} \gg 1$, we have $n_l \approx n_t$. Similarly, we have $T_t = (\langle t\rangle + \langle l\rangle)/\langle t\rangle$. Equation (2) then follows.

According to Eqs. (3) and (5), we have $T_l \approx 1$ for $\varepsilon \ll 1$. Also, we expect small variations in ε to have little influence on $\langle t\rangle$, as laminar motions are dominant for $p \gtrsim p_c$. The quantity that depends sensitively on ε is T_t, as can be seen from Eqs. (4) and (5). In this case we have $\beta \approx \alpha$.

To calculate $\langle l\rangle$, $\langle t\rangle$, T_l, and T_t numerically (with fixed ε and r) for discrete-time maps $\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n)$ that possess a periodic attractor of period p before the occurrence of intermittency, we first generate a typical trajectory. If, for some iteration i, the condition $\|\mathbf{x}_i - \mathbf{x}_c\| < r$ is satisfied, we set the length of the laminar phase l to be 1. We then examine $\mathbf{y}_{i+1} = \mathbf{F}^{(p)}(\mathbf{y}_i)$, where $\mathbf{y}_i = \mathbf{x}_i$, and $\mathbf{F}^{(p)}$ is the p-times iterated map of $\mathbf{F}$. If $\|\mathbf{y}_{i+1} - \mathbf{x}_c\| < r$, we update l to $l+1$. Otherwise, the length of this laminar phase is l. The average length $\langle l\rangle$ can then be obtained after a large number of laminar phases are accumulated. The average length of turbulent phase $\langle t\rangle$ can be computed similarly, where the turbulent phase is defined as $\|\mathbf{x}_i - \mathbf{x}_c\| > r$. The quantities T_l or T_t can be computed according to their definitions, where the p-times iterated map should be used. For continuous-time flows, these average quantities can be computed by using maps defined on a suitable Poincaré surface of section.

There are two major steps in our numerical computation of α for any given threshold r. The first is to find a reasonable scaling region, namely, a region in ε where $\langle l\rangle$ clearly shows a decrease as ε is increased. This can be done by extensive numerical trials. In all our computations, the scaling region spans over three orders of magnitude in ε, and there are at least ten points used in the linear fit on a logarithmic scale between $\langle l\rangle$ and ε for α to be extracted. Second, to compute $\langle l\rangle$ for a fixed ε in the scaling region, we generate a time series containing 1000 segments of the laminar phase and obtain $\langle l\rangle$ by averaging the length of these segments. All the scaling exponents are computed this way. We expect our results to be valid for small as well as for relatively large values of r. As we will show below, our systematic and extensive computations do reveal a strong dependence of the scaling exponent on the threshold r [9].

To provide numerical support for the dependence of the intermittency scaling exponent α on the threshold, we have examined the following four systems: (1) the classical logistic map [10] $x_{n+1} = a x_n (1 - x_n)$, where an intermittency occurs for $a < a_c = 1 + \sqrt{8}$; (2) the Hénon map [11] $(x,y) \to (a - x^2$

$+0.3y, x)$ for which an intermittency occurs for $a < a_c = 1.226\,617\,378\,076\ldots$; (3) the classical Lorenz system [12] $[\dot{x}, \dot{y}, \dot{z}] = [10(y-x), \rho x - y - xz, xy - 8/3]$, where an intermittency occurs for $\rho > \rho_c = 166.061\,492\,8\ldots$; and (4) the generalized Lorenz system [13]

$$[\dot{w}, \dot{x}, \dot{y}, \dot{z}] = [-(25\beta+10)(w-x), (28-35\beta)w + (29\beta-1)x - wy + z, -(8+\beta)y/3 + wx, -6w],$$

for which an intermittency occurs for $\beta < \beta_c = 0.079\,244\,927\ldots$.

Figure 1(a) shows, for the logistic map, the algebraic scaling of the average laminar-phase length $\langle l \rangle$ on a logarithmic scale for two different values of the threshold. Apparently, the scaling exponents for the two cases are quite different. Figure 1(b) shows the dependence of the scaling exponent α on the threshold. We observe that there is a continuous range of values of the exponent for the range of threshold values considered. The theoretical value of $\alpha = 1/2$ is achieved for sufficiently large threshold but there are fluctuations. Dependencies of the intermittency exponent on the threshold value for other three systems are shown in Fig. 2, where we observe that the fluctuations in α are quite large for continuous-time systems even after α begins to converge.

In our numerical experiments, we have observed that there exists a range of threshold values for which the average length $\langle t \rangle$ of the turbulent phase and the average recurrent time T_l to the laminar region are relatively constant. In such cases, Eq. (2) suggests that the average recurrent time to the turbulent region T_t obey the same scaling as the average laminar-phase length $\langle l \rangle$. Numerical calculations indeed point out an approximately linear relation between T_t and $\langle l \rangle$, as exemplified in Fig. 1(c) for the logistic map and in Fig. 3 for the other three systems. We remark that such an approximately proportional dependence does not hold unconditionally, rather, it is dependent upon the choice of ε and r. Only when this proportional relation holds is the theoretical value of $\alpha = 1/2$ attained, as can be seen from the results in Figs. 2 and 3.

In summary, we have revisited the problem of universal intermittency scaling in nonlinear dynamical systems by investigating the dependence of the scaling exponent on the choice of the threshold used to define laminar versus turbulent phases. Extensive numerical computations reveal that the exponent can depend strongly on the threshold. Insights into the dependence can be obtained by using the theory of recurrence. Our results suggest that proper caution should be exercised when claiming universal intermittency classes based on numerical or experimental scaling of the average length of the laminar phase with parameter variation.

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